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Non-embedding of non prime-power unitals with point-regular group

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Abstract

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Mathon (1987) and B. Bagchi and S. Bagchi have constructed a class of Steiner 2-designs, including some unitals, admitting a point-regular automorphism group. We show that any unital constructed by this method cannot be embedded in a projective plane π in such a way that the unital arises from a polarity and the point-regular group of the unital is induced by an automorphism group of π.

Introduction

A *unital* (or *unitary block design*) with parameter u is a 2-$(u^3+1, u+1, 1)$ design. It is well known that the absolute points and non-absolute lines of a unitary polarity of $PG(2, u^2)$ form a unital with parameter u. (This, of course, implies the existence of unitals with parameter u for any prime power u.)

In 1946 Baer [1] showed that if θ was a polarity of a finite projective plane of order n with $a(\theta)$ absolute points, then $n+1 \leqslant a(\theta)$. In 1970 Seib [7] improved this to show $n+1 \leqslant a(\theta) \leqslant n^{3/2}+1$. Furthermore, Seib showed that if $a(\theta) = n^{3/2}+1$, then the absolute points and non-absolute lines form a unital with parameter $u = \sqrt{n}$.

By conducting a systematic study of polarities of finite projective planes Ganley [3–4] discovered many (mutually non-isomorphic) examples of unitals. However, because they come from polarities of translation planes, all of Ganley's unitals had parameters which were prime powers. Indeed, it was widely conjectured that unitals could only exist for parameters which were prime powers.

In a recent paper Mathon [6] has constructed a class of cyclic Steiner 2-designs, including a unital with parameter 6. In another recent paper Bagchi and Bagchi [2] have given a construction for Steiner 2-designs admitting a point-regular group. Their construction includes the cyclic designs of Mathon which are unitals. The unital with parameter 6 is the first example of a unital with non prime-power parameter. It seems natural to ask the following question: Can we embed the unitals with parameter u arising from these constructions in a projective plane of order u^2 (as the absolute points and non-absolute lines of a polarity)? This is, of course, an exciting concept as it would give the first finite projective planes of non prime-power order.

In this short note we consider a special case of the problem, and prove the following.

Theorem 1. *Let D be a design constructed as in* [2] *which is a unital $U = U(u)$. Suppose U can be embedded in a projective plane π of order u^2 in such a way that U arises from the absolute points and non-absolute lines of a polarity σ of π, and such that the point-regular automorphism group E is induced by an automorphism group of π. Then $u = 2$.*

Notation. For a prime power Q, let $\mathrm{GF}(Q)$ denote the Galois field of order Q, G_Q the multiplicative subgroup of $\mathrm{GF}(Q)$, and $G_Q(M)$ the unique subgroup of order M in G_Q, where $M \mid Q-1$. Let $\overline{G_Q(M)}$ be $G_Q(M) \cup \{0\}$. For prime powers $P = p^h$ and $Q = q^i$, let $E(PQ)$ be the direct product of h copies of the cyclic group of order p, and of i copies of the cyclic group of order q.

The construction. Let P, Q be odd prime powers such that $P-1 \mid Q-1$. Let f be an epimorphism

$$f : G_Q(P-1) \to G_P\left(\frac{P-1}{2}\right).$$

Extend f by defining $f(0) = 0$ so

$$f : \overline{G_Q(P-1)} \to \overline{G_P\left(\frac{P-1}{2}\right)}.$$

Define t to be the largest divisor of $P-1$, relatively prime to $(Q-1)/(P-1)$, and γ to be a generator of $G_Q((Q-1)/t)$. Let $X = \mathrm{GF}(P) \times \mathrm{GF}(Q)$ be the ring with component-wise operations. For $x \in X$, $A \subseteq X$, let xA, $x + A$ denote the multiplicative and additive translates of A. Consider the following subsets of X:

$$A_0 = \{(f(x), x) \mid x \in \overline{G_Q(P-1)}\},$$

$$A_j = (1, \gamma^j)A_0 \quad 0 \leq j < \frac{Q-1}{P-1},$$

$$A_\infty = \mathrm{GF}(P) \times \{0\}.$$

Define an incidence structure D whose points are the elements of X, and whose blocks are all the additive translates of A_∞ and A_j, $0 \leq j < (Q-1)/(P-1)$. Each A_i $(i = 0, 1, \ldots, (Q-1)/(P-1)-1, \infty)$ has P points and so every block has P points. The number of blocks is

$$\frac{Q-1}{P-1} \cdot PQ + 1 \cdot Q.$$

D is a 1-design. To show D is a 2-design we need to show that any two points occur in exactly one block. It is sufficient to show that any two points occur in at most one block. For any two points $B_1 = (D_1, E_1)$, $B_2 = (D_2, E_2)$, B_1 and B_2 are on a translate of A_∞ if and only if $E_1 = E_2$. Further, if $E_1 = E_2$ then B_1 and B_2 are not both on a translate of A_i $(0 \leq i < (Q-1)/(P-1))$.

Consider B_1 and B_2 with $E_1 \neq E_2$, on $A_j + (T, W)$. Then

$$(D_1, E_1) = (1, \gamma^j)(f(z), z) + (T, W),$$

$$(D_2, E_2) = (1, \gamma^j)(f(w), w) + (T, W), \quad \text{with } z, w \in G_Q(P-1).$$

Hence

$$(D_1 - D_2, E_1 - E_2) = (f(z) - f(w), \gamma^j(z - w)).$$

For $y \in \mathrm{GF}(P)$ define

$$D_y = \{z - w \mid z, w \in G_Q(P-1), z \neq w, f(z) - f(w) = y\}. \qquad (*)$$

D_y is the set of second co-ordinates for the "within set differences" of A_0, with first co-ordinate equal to y. If B_1 and B_2 are on two blocks $A_j + (T, W)$ and $A_k + (U, V)$, then

$$\begin{aligned}(D_1 - D_2, E_1 - E_2) &= (f(z) - f(w), \gamma^j(z - w)) \\ &= (f(u) - f(v), \gamma^k(u - v)), \quad \text{with } z, w, u, v \in G_Q(P-1),\end{aligned}$$

so

$$\gamma^j A_y \cap \gamma^k A_y \neq \emptyset, \quad \text{where } y = D_1 - D_2.$$

Thus, to show that two points are on at most one block, it is sufficient to show D_y consists of $P-1$ distinct elements, and the sets $\gamma^j D_y$ are pairwise disjoint. Bagchi and Bagchi showed that their construction gave a 2-design under certain conditions, in the following.

Theorem 2 (Bagchi and Bagchi [2, Theorem 1]). *Let P and Q be odd prime powers such that $P-1 \mid Q-1$. If $P \equiv 1 \bmod 4$ then fix a non-square y_0 in* GF(P). *Suppose there is an epimorphism $f: G_Q(P-1) \to G_P((P-1)/2)$ for which D_y (defined in $(*)$) satisfies the following conditions for $y = 1$ if $P \equiv 3 \bmod 4$, and for $y = 1$, y_0 if $P \equiv 1 \bmod 4$:*

(a) *D_y consists of $P-1$ distinct elements.*

(b) *Whenever two elements of* D_y *belong to the same coset of* $G_Q((Q-1)/t)$, *they belong to the same coset of* $G_Q((P-1)/t)$.
Then the above construction yields a 2-$(PQ, P, 1)$ *design* D *on which* $E(PQ)$ *acts as a point-regular automorphism group.*

For $P \leqslant 11$ Bagchi and Bagchi have investigated these conditions showing that they are satisfied for many values of Q, and hence have constructed many designs.

Proof of Theorem 1. Assume that D is a unital $U = U(u)$. Then U is a 2-$(PQ, P, 1)$ design and also a 2-$(u^3+1, u+1, 1)$ design. Hence $P = u+1$, $Q = u^2 - u + 1$ with $u \geqslant 2$ since $P \leqslant Q$. E acts as a point-regular automorphism group and has $(Q-1)/(P-1)+1$ block orbits. There are $(Q-1)/(P-1)$ orbits of size PQ (corrresponding to the translates of each of the A_i), and one orbit of size Q (corresponding to the translates of A_∞).

Let (a, b) denote the greatest common factor of integers a and b. Then

$$(u^2 - u + 1, u+1) = ((u+1)(u-2)+3, u+1) = (3, u+1).$$

First, suppose $3 \mid u+1$. Then $P = 3^h$, $Q = 3^i$, $1 \leqslant h \leqslant i$. If $i > 1$ then $P = 3^h$ gives $u = 3^h - 1$. So $3^i = Q = (3^h-1)^2 - (3^h-1) + 1 = 3(3^{2h-1} - 3^h + 1)$. However,

$$3 \nmid 3^{2h-1} - 3^h + 1$$

and so the assumption $i > 1$ is false. If $i = 1$ then $h = 1$ and E has two orbits, one of size 9 and one size 3. U is the unique (classical) unital $U(2)$ and is isomorphic to the affine plane of order 3 [5].

Now assume $3 \nmid u+1$. Then $3 < P \leqslant Q$. Recall that t was defined to be the largest divisor of $P - 1 = u$, relatively prime to

$$\frac{Q-1}{P-1} = \frac{u^2-u}{u} = u - 1.$$

But as $(u, u-1) = 1$, $t = u$. Hence the block set is all additive translates of $A_0, A_1, \ldots, A_{u-2}, A_\infty$. E has u block orbits $\theta'_0, \ldots, \theta'_{u-2}, \theta'_\infty$. The orbits $\theta'_0, \ldots, \theta'_{u-2}$ are of size PQ and correspond to $A_0, \ldots, A_{u-2}$, and θ'_∞ is of size Q and corresponds to A_∞. E is semi-regular on the orbits $\theta'_0, \ldots, \theta'_{u-2}$. Consider elements $\alpha, \beta \in E$ such that the order of α is p and the order of β is q. By assumption, we can extend α and β to automorphisms $\bar{\alpha}, \bar{\beta}$ of π. Let $\theta_i = \theta'_i\sigma$, $(i = 0, \ldots, u-2, \infty)$. $\bar{\alpha}$ and $\bar{\beta}$ act in the same way on each θ_i, $(i = 0, \ldots, u-2, \infty)$ as they do on θ'_i since they commute with the polarity σ. As E is semi-regular on $\theta'_0, \ldots, \theta'_{u-2}$, so is $\langle\bar{\alpha}\rangle$. As $(P, Q) = 1$, $p \mid Q$ so $\bar{\alpha}$ fixes at least one line of θ'_∞ (and one point of θ_∞). But E is abelian, so α fixes all lines of θ'_∞ (and all points of θ_∞). On the other hand, $\langle\bar{\beta}\rangle$ acts semi-regularly on θ'_∞ and θ_∞.

We now consider the incidence matrix A of π. If $\bar{U}$ is the incidence matrix for the unital, then since we are assuming U arises from a polarity σ of π, we have

$$A = \begin{pmatrix} I & \bar{U} \\ \bar{U}^{\mathrm{T}} & B \end{pmatrix}$$

where $\bar{U}^{\mathrm{T}}$ represents the transpose of the matrix $\bar{U}$. I is the identity matrix of size PQ. A is symmetric and as σ has exactly u^3+1 absolute points, the entries on the leading diagonal, other than the first PQ entries, are all zero.

A may be partitioned into submatrices indexed by the point and line orbits

$$A = \begin{array}{cc} & \begin{array}{cccc} \theta'_0 & \cdots & \theta'_{u-2} & \theta'_\infty \end{array} \\ \begin{array}{c} \\ \theta_0 \\ \vdots \\ \theta_{u-2} \\ \theta_\infty \end{array} & \left(\begin{array}{c|ccc:c} I & & & & \\ \hline & & & & \\ & & & & \\ & & & & \\ \hdashline & & & & C \end{array}\right) \end{array}.$$

We have shown that the points of θ_∞ are exactly the fixed points of $\langle \bar{\alpha} \rangle$, and the lines of θ'_∞ are exactly the fixed lines of $\langle \bar{\alpha} \rangle$. So C is the incidence matrix of the fixed set of $\bar{\alpha}$. There are three cases to consider:

Case (1) *All the fixed points are collinear.*

Case (2) *The set of fixed points contains a triangle but no quadrangle.*

Case (3) *The set of fixed points contains a quadrangle, and hence is a projective plane.*

In Case (1) C has a column of ones, hence C has a one on the leading diagonal, a contradiction.

In Case (2) C is of the form

$$C = \left(\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 1 & & & & \\ 1 & & D & & \\ \vdots & & & & \\ 1 & & & & \end{array}\right),$$

D having exactly one one in every row. Consider the action of $\langle \bar{\beta} \rangle$ on the points of θ_∞. β does not fix the first row of C so some row of D has $Q-2$ ones. This is a contradiction as $Q > 3$.

In Case (3) C is the incidence matrix of a projective plane π'. Since A is symmetric, so is C. Thus C defines a polarity on π', with the number of absolute points being the trace of C. But the trace of C is zero, contradicting [5, p. 240, Lemma 12.3] which states that every polarity in a projective plane has at least one absolute point.

Thus we conclude that other than the unital $U(2)$, none of these point-regular unitals can be embedded in a projective plane π such that the unital arises from a polarity, and the group E is induced by an automorphism group of π.

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